Lecture Notes on Finite Difference Schemes
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Introduction

Many problems related to environment are often formulated in terms of differential equations. Most times the equations are quite complicated, and analytical solutions seem out of reach. However, in the modelling stage, we may use some assumptions which lead us to simpler equations. By analyzing properties of their solutions (analytical or numerical) give us insight of the underlying physical phenomena.

The whole process involves three important stages: modeling, analytical, and numerical approach. Analytical methods of solving differential equations are mostly covered in undergraduate courses on differential equations. Therefore, in this lecture notes we focus on the numerical approach with a bit of modeling. Finally, good understanding of the physical problems and numerical analysis will allow us to study more complicated problems.
Chapter 1

Decay problem

This chapter is resumed from the lecture notes by G.S. Stelling [4].

Consider the problem of pollutant concentration in a lake. A mixture of pollutant with concentrations \( c_i \) flows into the lake with a discharge \( Q_{in} \). Also, there is an outflow with discharge \( Q_{out} \). Let \( c(t) \) denotes the concentration of pollutant in the lake at time \( t \) and \( c(0) \) is the initial concentration. Volume balance dictates that the rate of change of lake volume equals to \( Q_{in} - Q_{out} \), explicitly

\[
\frac{dV}{dt} = Q_{in} - Q_{out}.
\]

Conservation of mass states that the rate of change of pollutant in the lake equals to the difference between the influx and out flux pollutant.

\[
\frac{d(cV)}{dt} = Q_{in}c_i - Q_{out}c.
\]

Two equations above can be combine to yield one equation below

\[
\frac{dc}{dt} = -\frac{Q_{in}}{V}(c - c_i).
\] (1.0.1)

Show this! Notice that the concentration of pollutant \( c(t) \) does not depend on \( c_{out} \). When \( Q_{in} = 0 \), the equation above simplify to

\[
\frac{dc}{dt} = \lambda c, \quad \text{with} \quad \lambda = -\frac{Q_{in}}{V}, \quad \text{which is a negative number.}
\] (1.0.2)

Analytical solution of (1.0.2) is

\[
c(t) = c_0e^{\lambda t}.
\] (1.0.3)

Parameter \(-1/\lambda = V/Q_{in}\) is called flushing time, relaxation time, age. It represents the ratio between the lake volume \( V \) and discharge \( Q_{in} \). This parameter denotes the time at which the concentration has reduced by a factor \( 1/e \). Through this simple ordinary differential equation, we will review several numerical integration method, including its accuracy, stability, and convergence.

1.1 Time integration scheme

The following are choices of difference equations to approximate the differential equation (1.0.2).

- Euler’s explicit
  \[
  \frac{c^{n+1} - c^n}{\Delta t} = \lambda c^n
  \]
  \[
  \tau_{\Delta t} = c_{it}^n \frac{\Delta t}{2} + O(\Delta t^2)
  \]

- Euler’s implicit
  \[
  \frac{c^{n+1} - c^n}{\Delta t} = \lambda c^{n+1}
  \]
  \[
  \tau_{\Delta t} = -c_{it}^n \frac{\Delta t}{2} + O(\Delta t^2)
  \]

- Trapezoidal rule
  \[
  \frac{c^{n+1} - c^n}{\Delta t} = \frac{1}{2}(c^{n+1} + c^n)
  \]
  \[
  \tau_{\Delta t} = -c_{itt}^n \frac{\Delta t^2}{12} + O(\Delta t^3)
  \]

- \( \theta \)-method
  \[
  \frac{c^{n+1} - c^n}{\Delta t} = \lambda (\theta c^{n+1} + (1 - \theta)c^n)
  \]
  \[
  \tau_{\Delta t} = \lambda \frac{\Delta t^2}{2} + O(\Delta t^3)
  \]
In the above equations the error \( \tau_{\Delta t} \) is defined as the difference between the differential equation and the approximate difference equations. The accuracy order of difference equations are determined from the order of \( \tau_{\Delta t} \). Task: Derive all the error above.

It is easy to check that the Euler explicit formula is equivalent with the recursion formula \( c^{n+1}/c^n = r_{\text{exp}} \), with

\[
   r_{\text{exp}} = (1 + \lambda \Delta t) .
\]

Similarly, the other formulas can be written as recursive formulas with ratio

\[
   r_{\text{imp}} = 1/(1 + \lambda \Delta t) ,
\]

\[
   r_{\text{trap}} = \frac{1 + \frac{1}{2} \lambda \Delta t}{1 - \frac{1}{2} \lambda \Delta t} .
\]

Exercise 1.1.1. Time integration scheme Consider \( c' = \lambda c \), with \( c(0) = 1 \). Use \( \lambda \Delta t = -0.1 \) to calculate \( c(t) \), for 20 time steps using the above numerical integrations: Euler’s explicit, implicit, and trapezoidal. Reproduce Figure 1.1.1 (left) that shows the numerical results together with the exact solution \( c_{\text{exact}}(t) = e^{\lambda t} \). It is shown that the numerical solutions from the explicit, implicit and trapezoidal schemes are all nicely approximate the analytical solution. As a second order scheme, the trapezoidal approximates better compare with the other two schemes. But if we repeat the same calculations using different values of \( \lambda \Delta t \), strange behaviour arise. Figure 1.1.1 (right) displays several computation results using \( \lambda \Delta t = -0.5, -1, -2, \) and \(-4\). Clearly, the computed results strongly depends on the choice of \( \lambda \Delta t \), and this issue is related to the concept of stability.

![Figure 1.1.1: Computed \( c(t) \) together with the exact solution, using (left) \( \lambda \Delta t = -0.1 \), (right) \( \lambda \Delta t = -0.5, -1, -2, \) and \(-4\).](image)

A numerical scheme is called stable if and only if the computed solution is finite at all times. In this problem, numerical solutions \( c^n, n = 0, 1, 2, \ldots \) compose a geometric series with ratios given as (1.1.1-1.1.3). Stability of the scheme directly comes from stability of the geometric series. For consistence and convergence aspects, readers may consult the great textbook [3].

Task: Derive the stability conditions for the three schemes: Euler explicit, Euler implicit, and trapezoidal schemes.
Exercise 1.1.2. Pollutant problem Suppose $Q_{out} = Q_{in} = Q$, and the volume of the mixture in the lake is $V$, a constant. Mass conservation principle will result in

$$V \frac{dc}{dt} = Q(c_{in} - c).$$

Suppose the concentration of inflow pollutant follows a harmonic function $c_{in} = \hat{C} \sin(\omega t + 1)$. Next, to reduce the number of parameters, we introduce the following non-dimensionless variables (scaling)

$$c' = \frac{c}{\hat{C}}, \quad t' = \omega t, \quad \Omega = \frac{\omega V}{Q}.$$

Variables with ‘ are non-dimensionless. The mass conservation becomes

$$\frac{dc'}{dt'} = \frac{1}{\Omega} (\sin t' + 1 - c')$$

with initial concentration $c'(0)$. We note that the equation in non-dimensionless variables only depends on one parameter only, which is $\Omega$. Parameter $\Omega$ is called buffer, and its interpretation will be clear when we compare $c_{in}(t)$ with $c(t)$ (assignment d).

1. Find the exact solution of (1.1.4). Determine initial concentration $c'(0)$ such that the solution does not contain homogeneous solution.

2. Implement explicit, implicit, and trapezoidal scheme to find approximate solution of (1.1.4).

3. Discuss stability, consistency, and convergence (optional) of the above schemes.

4. Calculate the concentration $c'(t)$ using initial concentration $c'(0)$ as obtained in a. Plot the numerical result against the analytical solution, and plot them together with the inflow $c_{in}(t)$. Compute for medium buffer $\Omega = 2$, with $\Delta t' = \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}$, large buffer $\Omega = 20$, with $\Delta t' = \frac{\pi}{8}$, and small buffer $\Omega = 0.2$, with $\Delta t' = \frac{\pi}{8}$. Interpret the result. Is stability condition of the schemes depends on $\Omega$? Give a reason.
Chapter 2

Harmonic oscillators

This chapter is resumed from J. Kämpf, *Ocean Modelling for Beginners*, Springer, 2009.

Harmonic oscillators occur widely in nature. When there is no damping, the equation for harmonic oscillator is
\[ u'' + \lambda^2 u = 0. \]
The analytical solution is a simple harmonic motion
\[ u(t) = A \cos \lambda (t - \phi), \]
which is an oscillation about the equilibrium point, with a constant amplitude \( A \) and a constant frequency \( \lambda \). In this section, we discuss the finite difference methods for solving a simple harmonic oscillator. It is important that our numerical scheme should be free from damping error.

2.1 Motion of a bouyant object

Density of seawater is not constant, instead it depends on temperature, salinity, and pressure. Near the surface, water is relatively warm, and hence its density is relatively low. In the deeper area, seawater density increases. It is often sufficient to approximate seawater density with a linear function as
\[ \rho_{\text{amb}}(z) = \rho_0 \left( 1 + \frac{N^2 g}{|z|} \right), \tag{2.1.1} \]
where \( \rho_0 \) is density at the surface, and \( z \) is the vertical axis. The approximation (2.1.1) is known as the Burnt-Väisälä frequency. An object with mass \( M_{\text{obj}} \), volume \( V_{\text{obj}} \) is released in a water column. Instead of full gravity force, it will experience a reduced gravity force, called bouyancy force. According to Archimedes’ principle, the resultant force is the difference between the object’s mass with the mass of fluid replaced by the object’s volume. When we neglect all frictions, the Newton second law for vertical acceleration read as
\[ M_{\text{obj}} \frac{dw}{dt} = -g(M_{\text{obj}} - M_{\text{amb}}), \]
or equivalently
\[ \frac{dw}{dt} = -g \frac{\rho_{\text{obj}} - \rho_{\text{amb}}}{\rho_{\text{obj}}}. \tag{2.1.2} \]
The buoyancy force on the right-hand side of the latter equation varies in magnitude and sign in dependence of the object’s location $z$. On the other hand, the location of the object $z$ changes owing to its vertical speed $w$ according to:

$$\frac{dz}{dt} = w. \quad (2.1.3)$$

Equations (2.1.2,2.1.3) are coupled with each other.

The proposed finite difference scheme for (2.1.2,2.1.3) are as follows

$$\frac{w^{n+1} - w^n}{\Delta t} = -g \left( \rho_{\text{obj}} - \rho_{\text{amb}}(z^n) \right) / \rho_{\text{obj}}, \quad (2.1.4)$$

$$\frac{z^{n+1} - z^n}{\Delta t} = w^{n+1}. \quad (2.1.5)$$

Argue that the above scheme is an explicit scheme.

Exercise 2.1.1. Assume the density in a 100 m deep water column increases linearly with depth. The surface density $10^{25}$ kg/m$^3$ and $N^2 = 10^{-4}$ s$^{-2}$, so that density increases to $10^{26}$ kg/m$^3$ at the bottom. An object with density $10^{25.5}$ kg/m$^3$ is released at a depth of say, 80 m.

1. Find this motion numerically using the finite difference scheme (2.1.4, 2.1.5). Make sure that your numerical oscillation has constant amplitude over a long period of computational time.

2. Introduce friction in the model by adding a term $-R|w|w$ on the right hand side of (2.1.2). Discretize this damping system using $-R|w^n|w^{n+1}$.

3. Find the analytical solution, and show that it represents the oscillation motion of the bouyant object in the stratified water column. Hint: Eliminate $w$ from (2.1.2,2.1.3), and formulate an orde-2 ordinary differential equation in terms of $z$.

### 2.2 Coriolis force

We all live on a rotating earth. Motions of wind and currents we observe in daily life are motions on a rotating earth. Naturally, governing equations of geophysical fluid motion are written in a rotating frame of reference.

A frame that rotates according to earth rotation. If the equations of motion are written in this rotating frame, there appears additional terms due to Coriolis force (which is a fictitious force). Due to Coriolis effect, wind moves spiral to the right (clockwise) in the Northern Hemisphere, and spiral to the left (anticlockwise) in the Southern part, and in the equator area there is no Coriolis effect, as illustrated in the next figure.

Let $\mathbf{r}(t) = (x(t), y(t))$ denotes the position of a particle on a rotating earth, and $\mathbf{v} = (u(t), v(t))$ denotes its velocity. In a rotating frame of reference, when other forces are absence except Coriolis, movement of the particle is governed by

$$\partial_t u - f v = 0, \quad \partial_t v + f u = 0, \quad (2.2.1)$$

with $f = 2\Omega_r \sin \phi$ is the Coriolis parameter, $\Omega_r = 2\pi/T$ the rotation rate, $T$ the rotation period, and $\phi$ the geographical latitude. The particle trajectory $(x(t), y(t))$ can then be calculated from

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v. \quad (2.2.2)$$

A particle initially at a position $x = 0$ m and $y = 5$ m is obstructed by an initial speed $u = 0.5$ m/s and $v = 0.5$ m/s. Under a rotating frame of reference at rate of $f$. Task: Show that its trajectory is a circle with radius $0.5 \frac{T}{f}$, given by

$$x(t) = \frac{0.5}{f} (\sin ft - \cos ft), \quad y(t) = \frac{0.5}{f} (\sin ft + \cos ft). \quad (2.2.3)$$
Our numerical simulation should be able to produce this perfect circle, which is not an obvious task. Implementing $\theta$–scheme for (2.2.1) will give us

\begin{align}
  u^{n+1} &= u^n + \Delta t f \left( (1 - \theta)v^n + \theta u^{n+1} \right), \\
  v^{n+1} &= v^n - \Delta t f \left( (1 - \theta)u^n + \theta v^{n+1} \right),
\end{align}

(2.2.4)

for $n = 0, 1, \cdots$ as time step index. When $\theta = 0$ the scheme (2.2.4) is explicit, if $\theta = 1$ the scheme is implicit, and $\theta = 1/2$ the scheme is known as trapezoidal or Crank-Nicolson scheme. Further, particle trajectory $(x(t_n), y(t_n))$ as a function of time can be calculated by applying the forward Euler method to (2.2.2).

**Exercise 2.2.1.** Conduct the eigenvalues analysis of the amplification matrix $A$ corresponds to (2.2.4), and deduce the following: explicit scheme is unconditionally unstable, whereas implicit and trapezoidal schemes are unconditionally stable. Moreover, show that matrix $A$ from the trapezoidal scheme has eigenvalues with spectral radius one, i.e. $|\lambda_{1,2}| = 1$. Hence, this scheme is free from numerical damping error.

Implement the implicit and trapezoidal schemes to simulate the particle trajectory. In order to speed up the computational time, just take a dummy frequency $f = -2 * \pi / 100$ sec$^{-1}$. (Note that Coriolis parameter corresponds to clockwise rotation with a period of 24 hours is $f = 2 * \pi / 3600$ sec$^{-1}$ = $-7.27 \times 10^{-5}$ sec$^{-1}$. Result form the implicit scheme are given in Figure 2.2.1(a). Even tough the implicit scheme is stable, we observe a trajectory spiralling inward and a gradual decrease in speed, which is certainly incorrect. On the contrary, the trapezoidal results as in Figure 2.2.1(b) show the correct results, in which particle’s trajectory forms a perfect circle!

![Figure 2.2.1: The parcel trajectory in a rotating frame. (a) Trajectory from the implicit Euler scheme. (b) Trajectory from trapezoidal scheme. The red point denotes the reference location in the fixed frame of reference.](image)

*Hint:* Explicit formulas for the trapezoidal scheme, equivalent to (2.2.4) with $\theta = 1/2$ is as follows

\begin{align}
  u^{n+1} &= \frac{((1 - 0.25\alpha^2)u^n + \alpha v^n)}{1 + 0.25\alpha^2}, \\
  v^{n+1} &= \frac{((1 - 0.25\alpha^2)v^n - \alpha u^n)}{1 + 0.25\alpha^2},
\end{align}

(2.2.5)

with $\alpha = f \Delta t$. 
Chapter 3

Shoreline development

This chapter is resumed from C.B. Vreugdenhil, *Computational Hydraulics, An Introduction*

A groyne is a hydraulic structure built along a seashore to control erosion. Commonly, a groyne is installed perpendicular or slightly oblique to the shoreline, see Figure 3.0.1. If waves approach the coastline obliquely, a wave driven flow along the coast is generated. On a sandy coast, the flow together with the stirring action of breaking waves causes a longshore sand transport, also known as littoral drift. The presence of a groyne interrupts the water flow and limit this sand movement. Apart from its function to control erosion, installment of groyne may lead to changes of shoreline. In this section, we consider how shoreline can develop due to the newly constructed breakwater of groyne type.

![Figure 3.0.1: (Left) A breakwater groyne, source: http://www.geograph.org.uk/photo/1643935 (Right) A diagram demonstrating longshore drift.](image)

Without going deeply into the physics, we can have insight of the consequences of constructing a groyne. The approach we adopt here is originally due to Pelnard-Considère. Let \( y(x, t) \) denotes the shoreline at time \( t \), see Figure 3.0.1 (right). Transport of sand is governed by the following balance of sand in the control interval \([x, x + \Delta x]\) as follows

\[
\text{Input - output} = \text{storage} \\
Q\big|_x \Delta t - Q\big|_{x+\Delta x} \Delta t = (y|_{t+\Delta t} - y|_t) a \Delta x, 
\]

where \( Q(m^3/s) \) denotes the sediment discharge, and \( a \) the water depth. When we divide both sides with \( \Delta x \Delta t \), and taking the limit \( \Delta x \to 0, \Delta t \to 0 \), the conservation of mass reads

\[
\frac{\partial Q}{\partial x} + a \frac{\partial y}{\partial t} = 0. 
\]  

(3.0.1)

An estimate discharge formula from the CERC (Coastal Engineering Research Center) reads

\[
Q = E \sin 2\phi, 
\]  

(3.0.2)
where $E(m^3/s)$ is a coefficient proportional to the incident wave energy flux, and $\phi$ is the angle between wave front and the local orientation of the coast line, hence 

$$\phi = \phi_0 - \frac{\partial y}{\partial x},$$

(3.0.3)

and $\phi_0$ is the direction of wave incidence, see Figure 3.0.1 (right). Substituting (3.0.2) and (3.0.3) into (3.0.1) gives us an equation for the coastline position in the form of diffusion equation below (please check!)

$$\frac{\partial y}{\partial t} - D \frac{\partial^2 y}{\partial x^2} = 0,$$

(3.0.4)

with the diffusion parameter $D = 2E/a$, that depends on the parameter $E$ which is difficult to determine. An estimate can be made as follows, an equilibrium sand transport $Q_0$ is suppose to be known from field measurements. Then from (3.0.2) we get $Q_0 \approx E \cdot 2\phi_0$ and we obtain

$$D = Q_0(\phi_0a)^{-1}.$$

(3.0.5)

For the initial condition, we just take the current coastline as $y(x, 0)$. For the boundary conditions, there are few considerations.

- Far away ($x \to \pm \infty$) there is no change, so either $y = 0$ or $\frac{\partial y}{\partial x} = 0$.
- At the breakwater, the sand flux $Q$ is zero. Since $E \neq 0$, it leads to $\phi = 0$ or equivalently $\phi_0 = \frac{\partial y}{\partial x}$.

### 3.1 Finite difference scheme for diffusion equation

We start with the simplest finite difference method for solving diffusion equation. Consider (3.0.4) at the partition point $(x_j, t_n)$, applying the forward time center space (FTCS) yield

$$\frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} = D \frac{u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n}}{\Delta x^2}$$

or

$$u_{j}^{n+1} = (1 - 2S)u_{j}^{n} + S(u_{j+1}^{n} + u_{j-1}^{n}),$$

with $S = D \frac{\Delta t}{(\Delta x)^2}$.

(3.1.1)

Prove the discrete equation (3.1.1), show that FTCS method has an accuracy of order $O(\Delta t, \Delta x^2)$.

![Figure 3.1.1: FTCS stencil for diffusion.](image)

### Stability

Here, we implement the von Neumann stability analysis, with the following steps. Substituting the ansatz $u_{j}^{n} = \rho^n e^{ik\Delta x}$ into (3.1.1) will give us

$$\rho = (1 - 2S) + S(e^{ik\Delta x} + e^{-ik\Delta x}) = 1 + 2S(cos k\Delta x - 1).$$
The finite difference is stable if and only if the modulus of a complex number $u^n_j$ is less than one, or $|\rho| \leq 1$. Some manipulation will give us a simpler condition

$$|1 + 2S(\cos k\Delta x - 1)| \leq 1 \iff 0 \leq 2S \sin^2 \left(\frac{k\Delta x}{2}\right) \leq 1,$$

please check! In order to guarantee that modulus is always less than one for all wave number $k$, we need to take $2S \leq \frac{1}{2}$. Hence, a necessary condition for stability of (3.1.1) is

$$S = D \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}. \quad (3.1.2)$$

In the implementation of (3.1.1), in order to get finite $u^n_j$, we need to take $\Delta x$ and $\Delta t$ such that they satisfy (3.1.2).

The following are other schemes commonly employed, those are backward time center space (BTCS), and Crank-Nicolson method

1. (BTCS) Derive the finite difference formulation resulting from implementing BTCS to (3.0.4), and show that BTCS is unconditionally stable.

2. (Crank-Nicolson) Consider (3.0.4) at the partition point $(x_j, t_{n+\frac{1}{2}})$. Implement FTCS and BTCS, each with half time differences, adding both will result in

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} = \frac{k}{2} \left( \frac{u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}}{(\Delta x)^2} + \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} \right). \quad (3.1.3)$$

Please check! Show that the scheme is unconditionally stable.

**Exercise 3.1.1. Shoreline change simulation** In this exercise, we need to employ the finite difference scheme to solve the diffusion equation (3.0.4) with the suitable boundary condition. Here you will see the effect of boundary condition, in fact the change of shoreline is dictated by the choice of boundary conditions.

For the exercises below use the following parameters $a = 18$ m, $\phi_0 = 0.2$ rad, $Q_0 = 1.5 \times 10^6$ m$^3$/year.

1. On a domain length $L = 10$ km, a groyne of length $b = 300$ m is constructed right in the middle. Initially, we have a straight coastline that corresponds to an initial condition $y(x, 0) = 0$. The breakwater intercepts the longshore sand transport, and hence the sand flux is zero, i.e. $\partial y / \partial x |_{x=L/2} = \phi_0$. However, the values of $y$ will be different on both sides of the breakwater. Actually, if no sand passes the breakwater the left and right regions are completely decoupled. Implement the FTCS method using $\Delta x = 0.25$ km, $\Delta t = 0.06$ year to simulate the computed coastline after one year.

2. Repeat the above exercise for a groyne field that consists of three groynes, each are separated by 5 km, on a coast that extend up to 20 km long.
Bibliography


